Random representations of the symmetric group

I. Notation & statements of the problem

1. Rep. th. of finite groups

Defⁿ A repⁿ (ρ, V) of a finite gp G is a pair:

- V is a $\mathbb{C}$-f.d. v.sp.
- ρ is a group morphism $G \longrightarrow GL(V)$

character : $\chi^\rho : G \longrightarrow \mathbb{C}$

$$g \longmapsto Tr(\rho(g))$$

Thm • Every repⁿ is a direct sum of irred rep^s.

• The set of irred rep^s is finite (denoted by $\mathcal{Y}_G$)

Not $\lambda \in \mathcal{Y}_G$, ρ^λ, V^λ, χ^λ, $\hat{\chi}^\lambda := \dfrac{\chi^\lambda}{\dim(V^\lambda)}$
std elt

2. Planchorel measure

Defⁿ The Planchorel measure is the probability measure on $\mathcal{Y}_G$ defined by $P(\lambda) = \dfrac{(\dim V^\lambda)^2}{|G|}$

Rk $\sum_{\lambda \in \mathcal{Y}_G} P(\lambda) = 1$, because $\mathbb{C}[G] = \bigoplus_{\lambda \in \mathcal{Y}_G} V_\lambda^{\dim(V_\lambda)}$

Prop. Fix $g \in G$. $E(\lambda \longmapsto \hat{\chi}^\lambda(g)) = \begin{cases} 1 & \text{if } g = 1_G \\ 0 & \text{otherwise} \end{cases}$

<u>Proof</u> $\mathbb{E}(\hat{\chi}^{\cdot}(g)) = \sum_{\lambda \in \mathbb{Y}_G} \frac{(\dim V_\lambda)^2}{|G|} \frac{\chi^\lambda(g)}{\dim V_\lambda} = \frac{1}{|G|} \sum_{\lambda \in \mathbb{Y}_G} (\dim V_\lambda) \operatorname{tr}_{V_\lambda}(\rho^\lambda(g))$

$$\mathbb{E}(\hat{\chi}^{\cdot}(g)) = \frac{1}{|G|} \operatorname{tr}_{\mathbb{C}[G]} \underset{\substack{\uparrow \\ \text{left} \\ \text{reg repr}^n}}{\rho(g)} = \begin{cases} 1 & \text{if } g = 1_G \\ 0 & \text{otherwise} \end{cases}$$

3. Our problem

Consider some sequence of groups $(G_n)_{n \geq 1}$

Assume $G_1 \subseteq G_2 \subseteq \dots$

For each n, take $\lambda^{(n)}$ random in $\mathbb{Y}_{G_n}$ (Pl measure)

<u>Question</u>: Asymptotic behaviour of $\lambda^{(n)}$?

LLN? (Law of large numbers)

CLT? (Central limit thm)

<u>Related question</u>: Fix $g \in G_k$ (for some $k \geq 1$)

Asymptotic behaviour of $\chi^{\lambda^{(n)}}(g)$?

4. Symmetric group

$G_n = S_n$

$\mathbb{Y}_{S_n} = \{\text{partitions of } n\}$

$\rightarrow$ plenty of ways to compute the characters $\chi^\lambda(g)$

For partitions, there is a natural way to define "convergence".

II. Moment method

1. Preliminaries : group algebra and its center

$\mathbb{C}[G] = \{ \sum_{g \in G} c_g g \ , c_g \in \mathbb{C} \}$

$Z(\mathbb{C}[G]) = \text{Span}\,(K_\mu)_{\mu \text{ conj. classes of } G}$

where $K_\mu = \sum_{\substack{g \text{ in conj. class} \\ \text{indexed by } \mu}} g$

Repr $\longleftrightarrow$ Morph. $\mathbb{C}[G] \longrightarrow \text{End}(V)$
(of unital alg.)

$\rho^\lambda, \chi^\lambda, \hat{\chi}^\lambda$ ext. by linearity

<u>Lemma</u> (Schur) $\lambda \in \mathcal{Y}_G$, $x \in Z(\mathbb{C}[G])$. Then $\rho^\lambda(x) = \hat{\chi}^\lambda(x)\, \mathrm{id}_{V^\lambda}$

<u>Cor.</u> $\lambda \in \mathcal{Y}_G$, $x \in Z(\mathbb{C}[G])$, $y \in \mathbb{C}[G]$. Then $\hat{\chi}^\lambda(xy) = \hat{\chi}^\lambda(x)\, \hat{\chi}^\lambda(y)$.

<u>Proof</u> $\rho^\lambda(xy) = \rho^\lambda(x)\,\rho^\lambda(y) = \hat{\chi}^\lambda(x)\,\rho^\lambda(y)$

2. Moments of $\hat{\chi}^\cdot((1,2))$

$\mathbb{E}_n(\hat{\chi}^\cdot((1,2))) = 0$

$\mathbb{E}_n[\hat{\chi}^\cdot((1,2))^2] = ?$

Idea: Work in the centre

$\hat{\chi}^\lambda((1,2)) = \frac{2}{n(n-1)}\, \hat{\chi}^\lambda\Big(\sum_{i<j} (i,j)\Big)$ for any $\lambda \in \mathcal{Y}_{S_n}$ $\Big(\forall\, i<j \;\; \hat{\chi}^\lambda((i,j)) = \hat{\chi}^\lambda((1,2)) \Big)$

$$\hat{\chi}^\lambda(K_{(2,1^{n-2})})^2 = \hat{\chi}^\lambda(K_{(2,1^{n-2})}^2)$$

$$K_{(2,1^{n-2})}^2 = \sum_{\substack{i<j\\k<l}} (i,j)(k,l) = \sum_{\substack{i<j,k<l\\ i,j,k,l \text{ distinct}}} (i,j)(k,l) + \sum_{\substack{i<j,k<l\\ i=k, j\neq l}} \underbrace{(i,j)(k,l)}_{(i,l,j)}$$

$$= 2K_{(2,2,1^{n-4})} + K_{(3,1^{n-3})} + \cdots$$

$$= 2K_{(2,2,1^{n-4})} + 3K_{(3,1^{n-3})} + \frac{n(n-1)}{2} K_{(1^n)}$$

$$\Rightarrow \mathbb{E}_n(\hat{\chi}^\cdot((1,2))^2) = \left(\frac{2}{n(n-1)}\right)^2 \underset{\substack{\shortparallel\\ \mathbb{E}(\hat{\chi}^\cdot(K_{(2,1^{n-2})}^2))}}{\left(\frac{n(n-1)}{2}\right)} = \frac{2}{n(n-1)}$$

Rk Only coef of $id_n = K_{(1^n)}$ is important

$$\mathbb{E}(\hat{\chi}^\cdot((1,2))^m) = \left(\frac{2}{n(n-1)}\right)^m \mathbb{E}(\hat{\chi}^\cdot(K_{(2,1^{n-2})}^m))$$

$$= \left(\frac{2}{n(n-1)}\right)^m \#\text{ ways to write } id_n \text{ as a product of } m \text{ transp.}$$

if m is odd, this is 0

3. Asymptotics ($n \longrightarrow \infty$, m fixed)

$$F_n^{2m} = \frac{1}{2^{2m}} \#\{(i_1,j_1,\dots,i_{2m},j_{2m}) \text{ s.t. } i_1\neq j_1,\dots,i_{2m}\neq j_{2m},\ (i_1,j_1)\dots(i_{2m},j_{2m}) = id\}$$

Define $Ker(ii,jj)$ as the set part of $\{1,\bar{1},\dots,2m,\overline{2m}\}$ s.t. $i_s = i_t \Leftrightarrow s \underset{Ker(ii,jj)}{\sim} t$

$$\begin{array}{cccccc} 1 & \bar{1} & 2 & \bar{2} & 3 & \bar{3} \\ \end{array}$$

Ex $(2,1,2,3,4,5)$

$\{1,2\}, \{\bar{1}\}, \{\bar{2}\}, \{3\}, \{\bar{3}\}$

Conditions $i_1 \neq j_1, \dots, i_{2m} \neq j_{2m}$ depend only on $\mathrm{Ker}(ii,jj)$

$(i_1,j_1)\dots(i_{2m},j_{2m}) = id$

Call "good" a partition $\mathrm{Ker}(ii,jj)$ if the conditions are fulfilled

$$F_n^{2m} = \frac{1}{2^{2m}} \sum_{\substack{\pi \text{ good set part} \\ \text{of } \{1,\bar{1},\dots,2m,\overline{2m}\}}} \#\{(i_1,j_1,\dots,i_{2m},j_{2m}) \text{ s.t. } \mathrm{Ker}(ii,jj) = \pi\}$$

$\leftarrow$ index set does not depend on n

$$= \frac{1}{2^{2m}} \sum_{\pi \text{ good}} n(n-1)\dots(n-\underbrace{l(\pi)}_{\#\text{blocks of } \pi}+1)$$

What are good partitions of max length?

At most $l(\pi) = 2m$, because each number must appear twice

$l(\pi) = 2m$ is possible

$\Downarrow$

(lemma : each transposition appears twice)

$(i_1,j_1)\dots(i_{2m},j_{2m}) = id \rightsquigarrow \pi$ has no singletons

matching $(2m-1)!!$

either $i_s = i_t$ and $j_s = j_t$ $\quad$ If each number appears exactly twice

2^m or $i_s = j_t$ and $j_s = i_t$ $\quad$ and each transp. appears twice product is id

$$2^{2m} F_n^{2m} = 2^m (2m-1)!!\, n^{2m} \left(1 + O\left(\frac{1}{n}\right)\right)$$

$$\mathbb{E}(\hat{X}((1,2))^m) = \left(\frac{2}{n(n-1)}\right)^{2m} \frac{1}{2^{2m}} 2^m (2m-1)!!\, n^{2m}(1+o(1)) = \frac{2^m}{n^{2m}} (2m-1)!!\,(1+o(1))$$

$\frac{n}{\sqrt{2}} \hat{\chi}^{\cdot}((1,2)) \xrightarrow[\text{in moment, hence in distribution}]{} \mathcal{N}(0,1)$ [Kerov, 93-02], [Hora, 98]

Other permutations

$\frac{n^{k/2}}{\sqrt{k}} \hat{\chi}^{\cdot}((1,\ldots,k)) \longrightarrow \mathcal{N}(0,1)$ + asymp. ind.

for non-cycles, more complicated limit laws

III. Partial permutations and polynomial functions on Young diagrams

1. Partial permutations

Goal understand better

$K^2_{(2,1^{n-2})} = K_{(3,1^{n-3})} + 2\,K_{(2,2,1^{n-4})} + \frac{n(n-1)}{2} K_{(1^n)}$

$K^m_{(2,1^{n-2})} = (2m-1)!!\, 2^m n^{2m} K_{(1^n)} + \cdots$

Def^n A partial permutation is a pair (σ, d) where

- $d \subseteq \mathbb{N}$ finite set
- σ a permutation of d

$\downarrow$

σ a permutation of $\mathbb{N}$ with finite support

two types of fixed points

Not $(\{1,3,5\},(1\,5)) = (1\,5)(3)$

$\rightarrow$ algebra structure on $\mathcal{B}_\infty = \{$inf. lin. comb. of partial perm.$\}$

$(\sigma,d)(\sigma',d') = (\tilde{\sigma}\circ\tilde{\sigma}', d \cup d')$

extension by taking fixed points

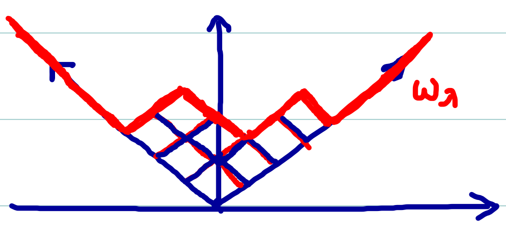
There exists an algebra morphism

$$\begin{aligned} B_\infty &\longrightarrow \mathbb{C}[S_n] \\ (\sigma, d) &\longmapsto \begin{cases} \vec{\sigma} & \text{if } d \subseteq \{1,\dots,n\} \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

IV. "Geometric" convergence of random partitions

1. What does it mean: "a sequence of partitions converges" $\lambda^{(n)}$ $(\lambda^{(n)} \vdash n)$?

Young diagrams in Russian convention :



(piecewise affine function

1-Lipschitz

$\omega_\lambda(x) = |x|$ outside a compact interval)

Def^n. A continuous Young diagram is a function $\omega : \mathbb{R} \to \mathbb{R}$

- 1 - Lipschitz
- $\omega(x) = |x|$ outside a compact interval

- "strong" conv. of cont. Young diagrams

$\omega_n \xrightarrow{\text{str.}} \omega$ iff $\|\omega_n - \omega\|_\infty \longrightarrow 0$

- "weak" conv. of cont. Young diagrams

$\omega_n \xrightarrow{\text{weak}} \omega$ iff $\int_{-\infty}^{\infty} \omega_n(x) - \omega(x) x^k \, dx \longrightarrow 0 \quad (|| \to \infty)$

for each fixed $k \geqslant 0$.

Renormalisation : $\lambda^{(n)} \vdash n$

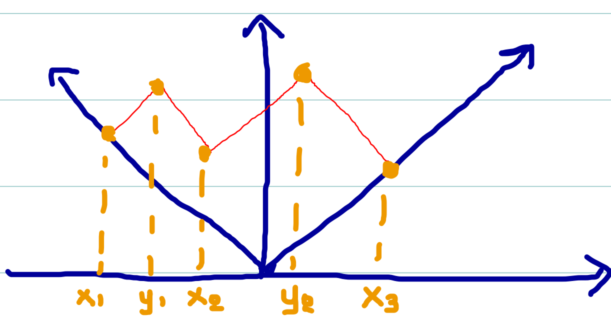
$$\omega_{\overline{\lambda^{(n)}}}(x) = \frac{1}{\sqrt{n}}\, \omega_{\lambda^{(n)}}(\sqrt{n}\, x)$$

Thm $\exists$ a deteministic function Ω s.t. $\omega_{\overline{\lambda^{(n)}}} \xrightarrow{\text{str}} \Omega$ in probability

r.part distributed with Pl. measure

2. A formula for character values

$\lambda^{(n)} \vdash n$



$$G_\lambda(z) = \frac{\prod_{i=1}^{m} (z-y_i)}{\prod_{i=1}^{m+1} (z-x_i)}$$

$$H_\lambda(z) = 1 / G_\lambda(z)$$

Thm: $n(n-1)\dots(n-k+1)\, \hat{\chi}^\lambda((1\dots k)) = -\frac{1}{k}[z^{-1}]\, \underbrace{H_\lambda(z)\dots H_\lambda(z-k+1)}_{\text{around } \infty}$

$* + c_1 z^{-1} + c_2 z^{-2} + \dots$

Define $M_k(\lambda)$ by $G_\lambda(z) = z^{-1} + \sum_{k\geq 2} M_k(\lambda)\, z^{-k-1}$

Then $n(n-1)\dots(n-k+1)\, \hat{\chi}^\lambda((1,\dots,k))$ is a polynomial in $M_2(\lambda), \dots, M_{k+1}(\lambda)$

coef of this poly do not depend on λ

Ex $n(n-1)(n-2)\,\hat{\chi}^\lambda((1,2,3)) = M_4(\lambda) - 4M_2^2(\lambda) + M_2(\lambda)$

$n(n-1)\dots(n-k+1)\,\hat{\chi}^\lambda((1,\dots,k)) = M_{k+1}(\lambda) + P_k(M_2(\lambda),\dots,M_k(\lambda))$

Triangular system : $M_{k+1}(\lambda) = n(n-1)\dots(n-k+1)\,\hat{\chi}^\lambda((1,\dots,k)) + P_k'\big((n)_i\,\hat{\chi}^\lambda((1,\dots,i))\big)_{i\le k}$

$$\log zG_\lambda(z) = -\sum \log\left(1-\frac{x_i}{z}\right) + \sum \log\left(1-\frac{y_i}{z}\right) \qquad zG_\lambda(z) = \frac{\prod(1-y_i/z)}{\prod(1-x_i/z)}$$

$$= +\sum_i\sum_{j\ge1}\frac{1}{j}\frac{x_i^j}{z^j} - \sum_i\sum_{j\ge1}\frac{1}{j}\frac{y_i^j}{z^j}$$

$$[z^{-j}]\log zG_\lambda(z) = \frac{1}{j}\left(\sum x_i^j - \sum y_i^j\right)$$

$$\underset{\text{comput.}}{=} \left(j-\frac{1}{2}\right)\int_{-\infty}^{+\infty}(\omega_\lambda(x)-|x|)\,x^{j-2}\,dx = \int_{-\infty}^{+\infty}(\omega_\lambda(x)-|x|)''\,\frac{x^j}{j(j-1)}\,dx$$

Define $p_k(\lambda) = \sum x_i^k - \sum y_i^k$

Claim : $\dfrac{p_{k+1}(\lambda)}{k+1} = n(n-1)\dots(n-k+1)\,\hat{\chi}^\lambda((1,\dots,k)) + P_k''\big((n)_i\,\hat{\chi}^\lambda((1,\dots,i))\big)_{1\le i\le k-1}$

First-order asympt. Interested in $\omega_{\overline{\lambda^{(n)}}}$

$$p_{k+1}(\overline{\lambda^{(n)}}) = \frac{1}{n^{k+1/2}}\,p_{k+1}(\lambda^{(n)}) = \frac{1}{n^{k+1/2}}\,P_k\big((n)_i\,\chi^\lambda((1,\dots,i))\big)_{1\le i\le k-1}$$

If $i\ge2$, $\dfrac{(n)_i\,\chi^\lambda((1,\dots,i))}{\sqrt{i}} \longrightarrow \mathcal{N}(0,1)$

$n^{i-1/2}\,\chi^\lambda((1,\dots,i)) \xrightarrow{(P)} 0$

Result : $\lim P_k(\lambda^{(n)}) \longrightarrow \begin{cases} \binom{2m}{m} & \text{if } k=2m \\ 0 & \text{otherwise} \end{cases}$ in proba

If one finds Ω s.t. $p_k(\Omega) = \begin{cases} \binom{2m}{m} & \text{if } k = 2m \\ 0 & \text{otherwise} \end{cases}$

then it means that $\lambda^{(n)} \xrightarrow[\text{strong}]{\text{weak}} \Omega$ in proba

<u>**Lemma:**</u> $P\left(\lambda_1^{(n)} \geq 3\sqrt{n} \text{ or } \lambda_1'^{(n)} \geq 3\sqrt{n}\right) = $ exp. small