QUANTUM RANDOM WALKS AND PITMAN THEOREM

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Harmonic oscillator

H Hilbert space, $\varepsilon_k, k = 0, 1, \ldots$ orthonormal basis

a^+, a^- creation and annihilation operators $a^+ = (a^-)^*$

$$[a^-, a^+] = I$$
$$a^+\varepsilon_k = \sqrt{k+1}\varepsilon_{k+1}$$
$$a^-\varepsilon_k = \sqrt{k}\varepsilon_{k-1}$$

"Heisenberg representation"

Probabilistic interpretation

$a^+ + a^-$=gaussian variable in state ε_0

$$\varepsilon_k = H_n(a^+ + a^-)\varepsilon_0$$

H_n =Hermite polynomial

Number operator

$a^+ a^- \varepsilon_k = k\varepsilon_k$ is the number operator
$a^+ a^- = \lim n - Z_n$

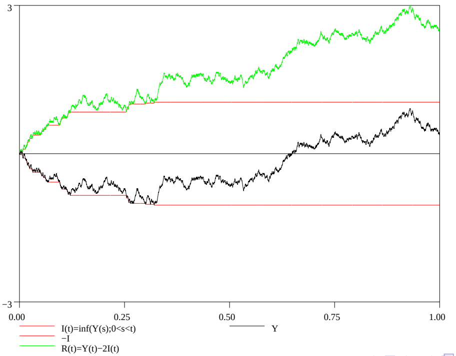
In the state ε_0, $a^+ a^-$ is the zero random variable

$\lambda(a^+ + a^+) + a^+ a^-$ has Poisson(λ^2) distribution.

cf Poisson as limit of binomial + recurrence relation for Charlier polynomials.

PITMAN THEOREM (1975)

$B_t; t \geq 0$ Brownian motion; $I_t = \inf_{0 \leq s \leq t} B_s$
$R_t = B_t - 2I_t; t \geq 0$ is distributed as the norm of a three dimensional
Brownian motion(=Bessel 3 process)
= eigenvalue process of a 2×2 hermitian brownian matrix



CONVERSE THEOREM

There is loss of information.

$R_t; t \geq 0$ =norm of a three dimensional Brownian motion

$x \in [0, 1]$ uniform random variable independent of R.

$$B_t = R_t - 2\inf(xR_T, \inf_{t \leq s \leq T} R_s); \quad t \in [0, T]$$

is a Brownian motion, and $R_t = B_t - 2I_t; t \geq 0$

$$X_i = \pm 1; \quad S_n = X_1 + X_2 + \ldots + X_n; \quad R_n = S_n - 2 \min_{0 \le k \le n} S_k$$

is a Markov chain(=discrete Bessel 3 process)

$$P(R_{n+1} = k+1 | R_n = k) = \frac{k+1}{2k}$$
$$P(R_{n+1} = k-1 | R_n = k) = \frac{k-1}{2k}$$

when $n \to \infty$
$S_{[nt]}/\sqrt{n} \to_{n\to\infty}$ Brownian motion
$R_{[nt]}/\sqrt{n} \to_{n\to\infty}$ norm of 3D-Brownian motion

EXTENSIONS

Gravner, Tracy, Widom (2001); $(B_1(t), \ldots, B_n(t))=$ n-dimensional Brownian motion

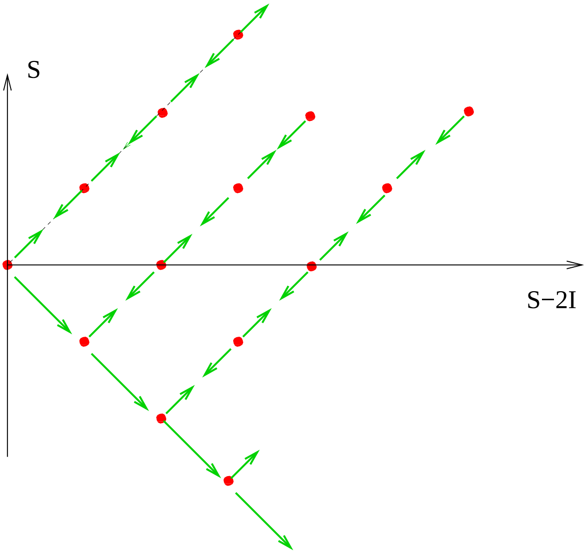
$$\lambda(t) = \sup_{1=t_n \geq t_{n-1} \geq \ldots \geq t_0 = 0} \sum_{i=1}^{n} (B_i(t_i) - B_i(t_{i-1}))$$

has the same distribution as the largest eigenvalue of a GUE matrix. Uses RSK correspondance

Generalized to a a representation of all eigenvalues by O'Connell and Yor (2002). Use queuing theory. and to Brownian motion on

the Lie algebra of a compact Lie group by Bougerol and Jeulin (2002). Uses Brownian motion on symmetric spaces.

PROOF OF PITMAN'S THEOREM



Quantization of head an tails game

$$X_n = \sum_{k=0}^{n-1} I^{\otimes k} \otimes x \otimes I^{\infty} \qquad Y_n = \sum_{k=0}^{n-1} I^{\otimes k} \otimes y \otimes I^{\infty}$$

$$Z_n = \sum_{k=0}^{n-1} I^{\otimes k} \otimes z \otimes I^{\infty}$$

in $M_2(\mathbb{C})^{\otimes\infty}$.

X_n, Y_n, Z_n define three simple random walks

$$[X_n, Y_n] = 2iZ_n$$

Let $R_n = \sqrt{X_n^2 + Y_n^2 + Z_n^2 + 1}$

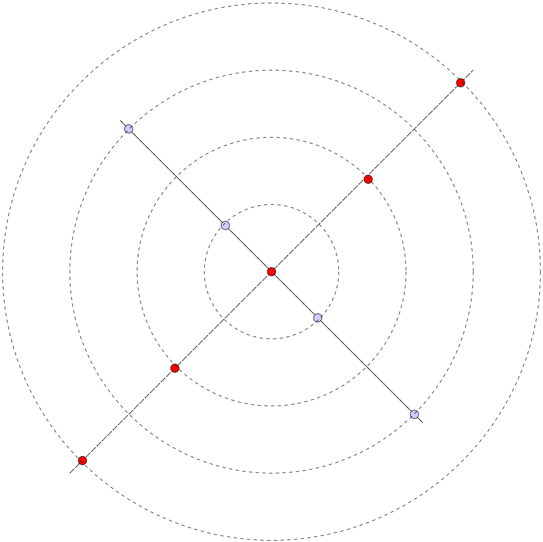
Lemma $[R_m, R_n] = 0$; R_n is a Markov chain with probability transitions

$$p(k, k+1) = \frac{k+1}{2k} \quad p(k, k-1) = \frac{k-1}{2k}$$

Proof: R_n corresponds to the Casimir operator.
Clebsch-Gordan formula for representations of $SU(2)$

$$[k] \otimes [2] = [k+1] \oplus [k-1]$$

We have defined a random walk with values in a noncommutative space $S\hat{U}(2)$



$A =$ group algebra of $SU(2)$
x, y, z=generators of Lie($SU(2)$)=coordinates on the space $S\hat{U}(2)$

$$[x, y] = 2iz$$

In each direction of space the coordinates take integer values.
One can measure the distance to origin using $\sqrt{x^2 + y^2 + z^2 + 1}$

$E =$ a set (e.g. Z^d)
Ω a probability space

A random variable with values in E: $X : \Omega \to E$

this gives an algebra morphism:

$$F(E) \to F(\Omega)$$

$$f \to f \circ X$$

We could drop the condition that the algebras are commutative

$A =$ group algebra of $SU(2)=$ Hopf algebra with coproduct

$$\Delta : A \to A \otimes A$$

$$\Delta(x) = x \otimes I + I \otimes x$$

$j_n : A \to M_2(\mathbb{C})^{\otimes\infty}$=$n$-fold tensor product of 2-dimensional representations for $n = 1, 2, ...$ form a quantum Bernoulli random walk

the quantum Bernoulli walk is a Markov chain with Markov operator

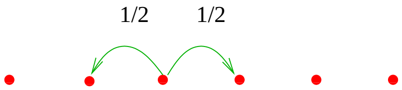
$$P : A \to A$$

$$P = Id \otimes Tr_2(./2) o \Delta$$

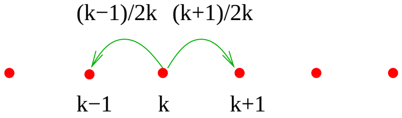
RESTRICTIONS

We can restrict the Markov operator P to commutative subalgebras:

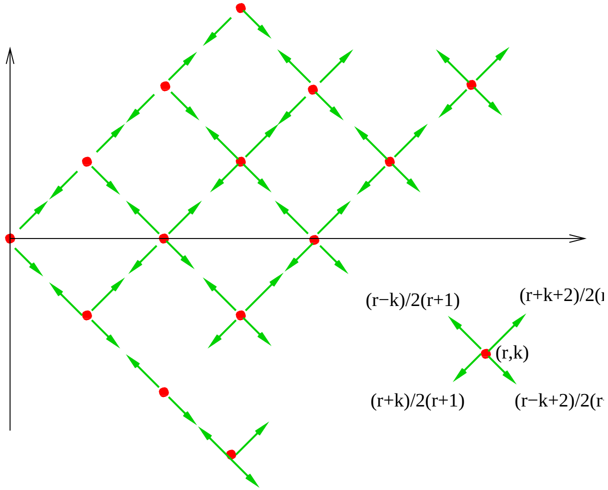
One parameter subgroup: Bernoulli random walk



Center: "discrete Bessel process"

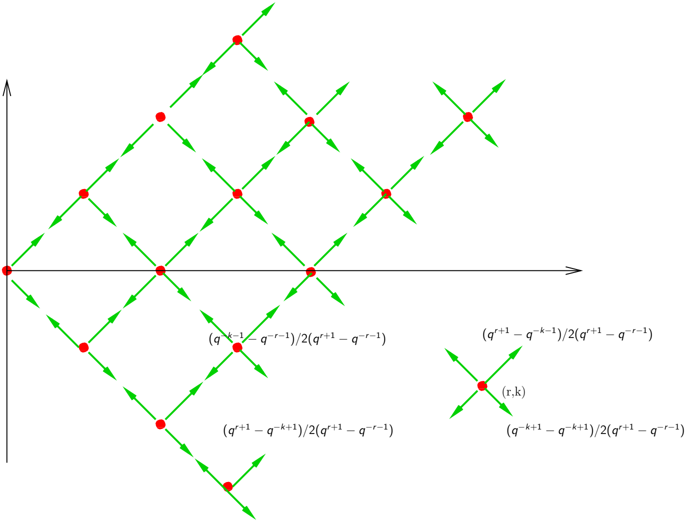


Maximal abelian subalgebra generated by the center and a one parameter subgroup



Kashiwara's crystallization

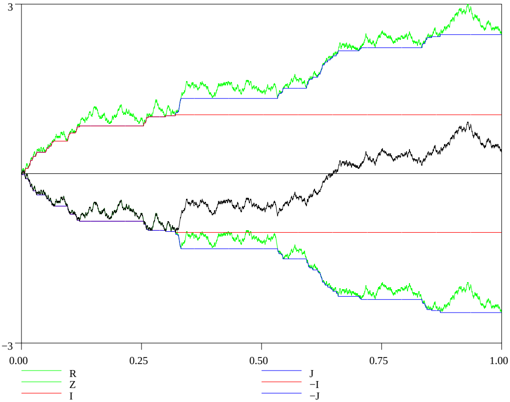
Replace $SU(2)$ by $SU_q(2)$ then



Let $q \to 0$ then one obtains Pitman's theorem.
cf Littelmann path model.

PITMAN OPERATORS

$$Y : [0, T] \to \mathbf{R}, \qquad Y(0) = 0$$



$$PY(t) = Y(t) - 2\inf_{0\leq s\leq t} Y(s)$$

For all t one has $PY(t) \geq 0$, in particular $PPY = PY$.

MULTIDIMENSIONAL PITMAN OPERATORS

V=real vector space, $\alpha \in V$, $\alpha^\vee \in V^*$ $\alpha^\vee(\alpha) = 2$.

$$P_\alpha Y(t) = Y(t) - \inf_{0 \le s \le t} \alpha^\vee(Y(s))\alpha$$

$$P_\alpha P_\alpha Y = P_\alpha Y$$

COMPOSITION OF PITMAN OPERATORS

$\alpha, \alpha^\vee, \beta, \beta^\vee$ satisfy $\alpha^\vee(\beta) = \beta^\vee(\alpha) = -2\cos\theta$ and $\theta \leq \frac{\pi}{n}$

$$(n \text{ terms}) \qquad P_\alpha P_\beta P_\alpha \dots Y(t) =$$

$$Y(t) - \inf_{t \geq s_1 \geq \dots \geq s_n \geq 0} A(s_1, \dots, s_n)\alpha - \inf_{t \geq s_1 \geq \dots \geq s_{n-1} \geq 0} B(s_1, \dots, s_{n-1})\beta$$

with

$$A(s_1, .., s_n) = \frac{\sin\theta}{\sin\theta}\alpha^\vee(Y(s_1)) + \frac{\sin 2\theta}{\sin\theta}\beta^\vee(Y(s_2)) + \frac{\sin 3\theta}{\sin\theta}\alpha^\vee(Y(s_3)) + \dots$$

$$B(s_1, .., s_{n-1}) = \frac{\sin\theta}{\sin\theta}\beta^\vee(Y(s_1)) + \frac{\sin 2\theta}{\sin\theta}\alpha^\vee(Y(s_2)) + \frac{\sin 3\theta}{\sin\theta}\beta^\vee(Y(s_3)) + \dots$$

Braid relations

If $\theta = \pi/n$ then

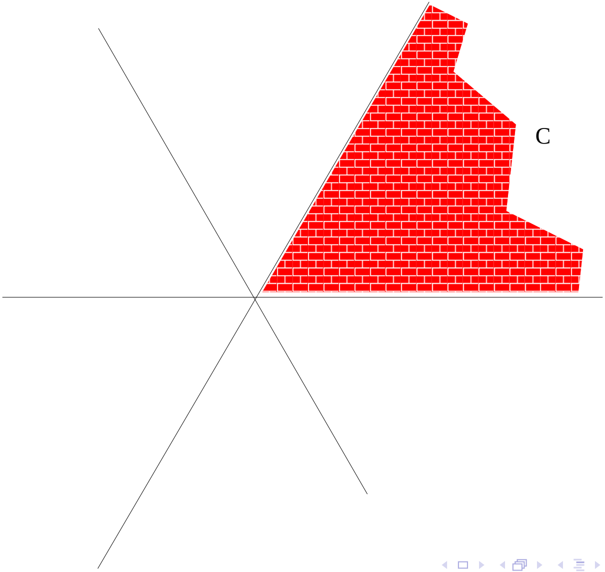
$$P_\alpha P_\beta P_\alpha \ldots = P_\beta P_\alpha P_\beta \ldots \qquad (n \text{ terms})$$

Corollary: Let (W, S)=Coxeter system on V and $\alpha, \alpha^\vee$=simple roots and coroots,
C=Weyl chamber. To each $s_\alpha \in S$ associate P_{s_α}. For each $w \in W$ with reduced decomposition $w = s_{\alpha_1} \ldots s_{\alpha_k}$ there exists

$$P_w = P_{s_{\alpha_1}} \ldots P_{s_{\alpha_k}}$$

If w_0=longest element then $P_{w_0}X$ takes values in C.

$W = S_3$



DOOB'S CONDITIONNED BROWNIAN MOTION

$$\Psi(x) = \prod_{\beta \in R_+} \beta(x)$$

is a positive harmonic function on C

$$p_t^W(x,y) = \sum_{w \in W} \varepsilon(w) p_t(x, w(y))$$

is the fundamental solution of Laplacian on W with Dirichlet boundary conditions
(=transition probabilities for Brownian motion killed at the boundary of C).

$$q_t(x,y) = \frac{\Psi(y)}{\Psi(x)} p_t^W(x,y)$$

are the transition probabilities of Brownian motion conditionned to stay in C.

Fact:

when $W = S_n$ (i.e. Weyl group of type A_{n-1} then Brownian motion conditionned to stay in C is the same as the motion of eigenvalues

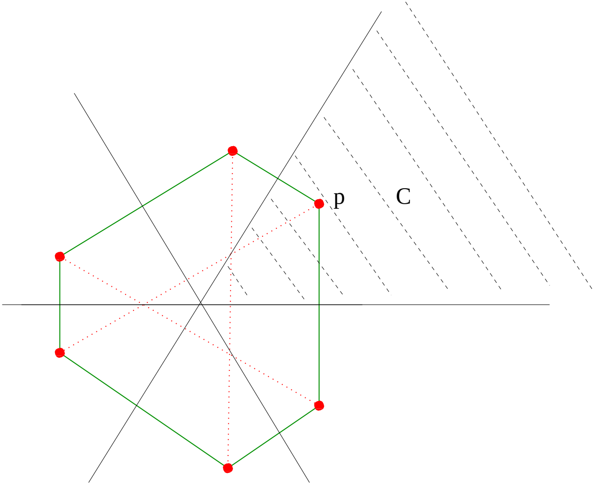
$$(\lambda_1(t)\lambda_2(t), \ldots, \lambda_n(t))$$

of a Brownian traceless hermitian matrix.

$$\big(M_{ij}(t)\big)$$

CONVERSE THEOREM

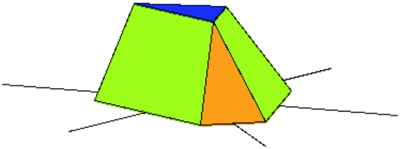
The conditional distribution of $X(t)$ knowing $P_{w_0}X(t) = p$ is the Duistermaat-Heckmann measure on the convex polytope with vertices $w(p); w \in W$.



Its Fourier transform is

$$\frac{1}{\prod_{\beta \in R} \beta(y)} \sum_{w \in W} \varepsilon(w) e^{i\langle p, y\rangle}$$

density is piecewise polynomial

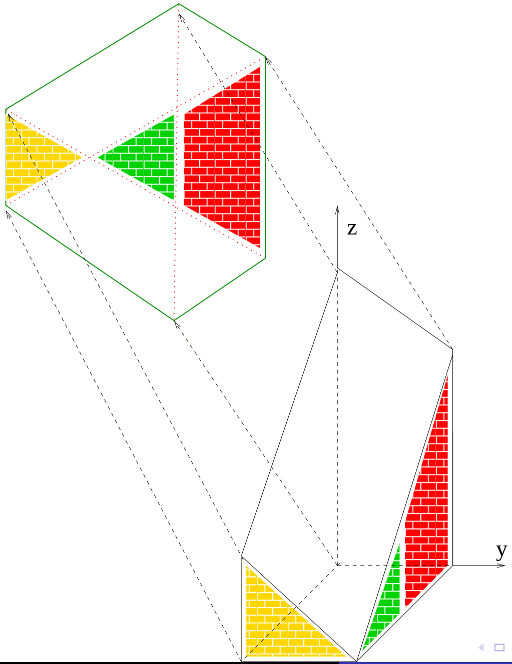


In order to recover X from $P_{w_0}X$ we need a positive real number x_i for each s_i in $P_{w_0} = P_{s_1} \dots P_{s_q}$.

Lemma Given $P_{w_0}X(t)$ the numbers $(x_1, \dots, x_q)$ belong to a certain convex polytope. Their distribution is the normalized Lebesgue measure on this polytope.

Cristallographic case: Berenstein-Zelevinsky polytopes

The Duistermaat-Heckman measure is the image of this measure by an affine map.



$0<x<a$

$0<y<b$

$0<z<(a-x)+(b-y)$

STURM-LIOUVILLE EQUATIONS

$$\varphi'' + q\varphi = \lambda\varphi$$

Let φ_0 be a > 0 solution on $[0, T]$. All other solutions are:

$$\varphi = a\varphi_0 + b\varphi_0 \int \frac{1}{\varphi_0^2(s)} ds$$

consider the maps

$$T_{a,b} : \varphi \mapsto a\varphi + b\varphi \int_0^t \frac{1}{\varphi^2(s)} ds$$

$$T_{a,b} T_{a',b'} = T_{aa',ab'+b/a'}$$

i.e. (almost) representation of

$$\begin{pmatrix} 1/a & 0 \\ b & a \end{pmatrix}$$

LAPLACE METHOD

$$\lim_{\varepsilon \to 0} \varepsilon \log \int_{x_0}^{x_1} \exp(-\frac{1}{\varepsilon} u(s)) ds = - \inf_{x_0 \le t \le x_1} u(t)$$

$$\begin{aligned}
&\lim_{\varepsilon \to 0} \varepsilon \log T_{e^{-x/\varepsilon},b}(\exp \frac{1}{\varepsilon} X(t)) \\
&\quad = \lim_{\varepsilon \to 0} \varepsilon \log \left(e^{-\frac{x}{\varepsilon} + \frac{1}{\varepsilon} X(t)} + b e^{\frac{1}{\varepsilon} X(t)} \int_t^T e^{-\frac{2}{\varepsilon} X(s)} ds \right) \\
&\quad = X(t) - 2 \inf_{0 \le s \le t} X(s) \wedge x \qquad (b > 0)
\end{aligned}$$

For $x = +\infty$ one gets Pitman operator.

MATRIX INTERPRETATION OF $T_{a,b}$

consider

$$\dot{M}(t) = \begin{pmatrix} \dot{X}(t) & 1 \\ 0 & -\dot{X}(t) \end{pmatrix} M(t) \qquad M(t) = \begin{pmatrix} e^{X(t)} & e^{X(t)} \int_0^t e^{-2X(s)} ds \\ 0 & e^{-X(t)} \end{pmatrix}$$

$$A \in GL(2, R) \qquad MA = [MA]_< [MA]_\geq$$

Gauss decomposition ($[\cdot]_<$=strictly lower triangular ; $[\cdot]_\geq$= upper triangular)

Lemma

$$\frac{d}{dt}[MA]_\geq(t) = \begin{pmatrix} \frac{d}{dt} T_A X(t) & 1 \\ 0 & -\frac{d}{dt} T_A X(t)) \end{pmatrix} [MA]_\geq$$

This gives an (almost)action $A \mapsto T_A$ of GL(2) on functions X.

Take $s = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

$$T_s X(t) = X(t) + \log \int_0^t e^{-2X(s)} ds$$

Laplace method $\rightarrow$ Pitman transform

$$D_\varepsilon \circ T_s X(t) \circ D_{\frac{1}{\varepsilon}} \rightarrow_{\varepsilon \rightarrow 0} PX(t) = X(t) - 2 \inf_{0 \leq s \leq t} X(s)$$

$$D_\varepsilon X(t) = \varepsilon X(t)$$

Higher rank

Consider root data $\alpha_i, \alpha_i^\vee, e_i, f_i, h_i$, et $t \mapsto X(t) \in \mathfrak{H}$ and the solution to

$$\dot{M}(t) = (\dot{X}(t) + N)M(t)$$

$N = \sum_i e_i$ (or more generally $\sum_i u_i e_i; u_i > 0$)

Lemma

$$\frac{d}{dt}[MA]_\geq(t) = (\frac{d}{dt} T_A X(t) + N)[MA]_\geq$$

Relies on $Ad_{n_-}(x)_\geq = x$ iff $x \in [span_i(e_i)]$

Laplace method $\to$ Pitman operators to T_{s_i} $s_i =$ simple reflections
Braid relations for Pitman operators follow from those of T_{s_i}

Application : a formula for generalized Pitman operators

Recall

$$(n \text{ termes}) \qquad P_\alpha P_\beta P_\alpha \dots Y(t) = Y(t) -$$

$$2 \inf_{t \geq s_1 \geq \dots \geq s_n \geq 0} \left[\frac{\sin\theta}{\sin\theta} \alpha^\vee(Y(s_1)) + \frac{\sin 2\theta}{\sin\theta} \beta^\vee(Y(s_2)) + \dots \right] \alpha -$$

$$2 \inf_{t \geq s_1 \geq \dots \geq s_{n-1} \geq 0} \left[\frac{\sin\theta}{\sin\theta} \beta^\vee(Y(s_1)) + \frac{\sin 2\theta}{\sin\theta} \alpha^\vee(Y(s_2)) + \dots \right] \beta$$

Analogous formula holds for any generalized Pitman operator

$X(t) \in H$, solution to

$$\dot{M}(t) = (\dot{X}(t) + N)M(t)$$

$$e^{X(t)} \sum_{k \geq 0} \sum_{i_1, \ldots, i_k} \left(\int_{t \geq t_1 \ldots \geq t_k \geq 0} e^{-\alpha_{i_1}^{\vee}(a(t_1)) - \ldots - \alpha_{i_k}^{\vee}(a(t_k))} dt_1 \ldots dt_k \right) e_{i_1} \ldots e_{i_k}$$

Laplace method ($P_w X = \lim D_\varepsilon T_w D_{1/\varepsilon}$):

$$P_w X(t) = X(t) - \sum_i \left(\inf_{\substack{j_1, \ldots, j_r \in S(\omega_i, w) \\ t \geq t_1 \ldots \geq t_k \geq 0}} \alpha_{j_1}(X(t_1)) + \ldots + \alpha_{j_r}(X(t_r)) \right) \alpha_i$$

ω_i = fondamental weights
$S(\omega_i, w)$ = set of $(j_1, \ldots, j_r)$ such that $\langle e_{i_1} \ldots e_{j_r} w v_{\omega_i}, v_{\omega_i} \rangle \neq 0$
("**i**-trails" of Berenstein Zelevinsky)