QUANTUM RANDOM WALKS AND PITMAN THEOREM

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A crash course on quantum mechanics

$H =$ (complex) Hilbert space

Observables=self-adjoint operators on H

unit vector $\varphi \in H$ (state of the system)
$+$ A observable
$\rightarrow$ probability measure

$$P(\lambda) = |\pi_\lambda \varphi|^2$$

$$\pi_\lambda = \text{orthogonal projection on eigenspace of } \lambda$$

P is supported on the spectrum of A.

Expectation of A is

$$\langle A\varphi, \varphi \rangle = Tr(A\pi_\varphi)$$

More generally: expectation of $f(A)$ is

$$\langle f(A)\varphi, \varphi \rangle = Tr(f(A)\pi_\varphi)$$

One can convexify: replace π_φ with a positive operator of trace 1.

$$E[f(A)] = Tr(\rho f(A))$$

Basic example

(Ω, F, P) probability space

$H = L^2(\Omega, F, P)$

x=real random variable

$$\begin{aligned} X_x : H &\to H \\ X_x(z) &= xz \end{aligned}$$

is a self-adjoint operator
Spectral theorem: any self-adjoint operator on a Hilbert space can be put in this form.

If $A_1, \dots, A_n$ commute $\to$ diagonalized simultaneously

Their joint distribution makes sense:

$$Tr(\rho f(A_1, \dots, A_n) = \int f(x_1, \dots, x_n) d\mu$$

for μ proba on $\mathbf{R}^n$

Spins

dim(H)=2

The space of observable has dimension 3

Pauli matrices give a basis

$$x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad y = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \quad z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

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In the state e_1,
X and Y are symmetric Bernoulli
$Z = 1$ a.s.

In the central state $Tr(.\frac{1}{2}Id)$ all three are symmetric Bernoulli.

By choosing state appropriately on can realize any Bernoulli distribution.

Quantization of head an tails game

in $M_2(\mathbb{C})^{\otimes\infty}$

$$X_n = \sum_{k=0}^{n-1} I^{\otimes k} \otimes x \otimes I^{\infty} \qquad Y_n = \sum_{k=0}^{n-1} I^{\otimes k} \otimes y \otimes I^{\infty}$$

$$Z_n = \sum_{k=0}^{n-1} I^{\otimes k} \otimes z \otimes I^{\infty}$$

$$[X_n, X_m] = [Y_n, Y_m] = [Z_n, Z_m] = 0$$

X_n, Y_n, Z_n define three simple random walks

$$[X_n, Y_n] = 2iZ_n$$

Quantum central limit theorem

In the state $e_1^{\otimes\infty}$

X_n and Y_n are symmetric Bernoulli
$Z_n = n$

In the state $Tr(.\frac{1}{2}Id)^{\otimes_i nfty}$

X_n Y_n and Z_n are symmetric Bernoulli there is a basis $\varepsilon_k, k = 0, 1, ...$ such that

$$\epsilon_0 = e_1^{\otimes\infty}$$

$$Z_n \varepsilon_k = n - 2k$$
$$(X_n + iY_n)\varepsilon_k = \sqrt{k(n-2k+2)}\varepsilon_{k-1}$$
$$(X_n - iY_n)\varepsilon_k = \sqrt{(k+1)(n-2k)}\varepsilon_{k+1}$$

In the limit

$$Z_n/n, X_n/\sqrt{n}, Y_n/\sqrt{n}$$

converge to *harmonic oscillator*

Harmonic oscillator

H Hilbert space, $\varepsilon_k, k = 0, 1, \dots$ orthonormal basis

a^+, a^- creation and annihilation operators $a^+ = (a^-)^*$

$$[a^-, a^+] = I$$

$$a^+\varepsilon_k = \sqrt{k+1}\varepsilon_{k+1}$$

$$a^-\varepsilon_k = \sqrt{k}\varepsilon_{k-1}$$

"Heisenberg representation"

Probabilistic interpretation

$a^+ + a^-$=gaussian variable in state ε_0

$$\varepsilon_k = H_n(a^+ + a^-)$$

H_n =Hermite polynomial

Number operator

$a^+a^-\varepsilon_k = k\varepsilon_k$ is the number operator
$a^+a^- = n - \lim Z_n$

In the state ε_0, a^+a^- is the zero random variable

$\lambda(a^+ + a^+) + a^+a^-$ has Poisson(λ^2) distribution.

cf Poisson as limit of binomial + recurrence relation for Charlier polynomials.

Fock space

H complex Hilbert space $H^{o\,n}$=symmetric tensor powers

$\mathcal{F}(H) = \oplus_n H^{o\,n}$ is the Fock space

$$a_h^+(x_1ox_2o\dots ox_n) = hox_1ox_2o\dots ox_n$$

$$a^-h = (a_h^+)^*$$